

* FIRST order first degree D.E.

* Derivative Form:

$$a_1(x)y' + a_2(x)y = g(x)$$

$$\Rightarrow \frac{dy}{dx} + P(x)y = Q(x)$$

General form: $F^n(y', y, x)$

* Method of Solution

II Separation of Variable:-

بالفصلين بالمتغيرين

$$x dx + y dy = 0$$

* Solve The D.E.

ex: $\frac{dy}{dx} = \frac{y}{x^2}$

$$\frac{dy}{y} = \frac{dx}{x^2} \Rightarrow \int \frac{1}{y} dx = \int \frac{1}{x^2} dx$$

$$\ln y = \frac{-1}{x} + C \Rightarrow C = \ln A$$

$$\ln y = \frac{-1}{x} + \ln A \quad \ln y - \ln A = \frac{-1}{x}$$

$$\therefore y = A e^{-\frac{1}{x}}$$

$$y = A e^{-\frac{1}{x}} \rightarrow \textcircled{1}$$

$$\text{ex}_2 :- y' = \frac{x+1}{y^4+1}$$

$$\therefore \int (y^4+1) dy = \int (x+1) dx$$

$$\therefore \frac{y^5}{5} + y = \frac{x^2}{2} + x + C$$

$$\text{ex}_3 :- (x^2+4) \frac{dy}{dx} = xy$$

$$\therefore \int \frac{1}{y} dy = \int \frac{x}{x^2+4} dx$$

$$\ln y = \frac{1}{2} \ln(x^2+4) + C \Rightarrow \ln A$$

$$\therefore \ln y = \ln(x^2+4)^{1/2} + \ln A$$

$$y = A \sqrt{x^2+4}$$

$$\text{ex}_4 :- xy dx + e^{-x^2} (y^2-1) dy = 0 \quad \div (y \cdot e^{-x^2})$$

$$\int e^{-x^2} x dx + \int \left(\frac{y^2-1}{y} \right) dy = 0$$

$$\frac{1}{2} \int 2x e^{-x^2} dx + \int \left(y - \frac{1}{y} \right) dy = 0$$

$$\frac{1}{2} e^{-x^2} + \frac{y^2}{2} - \ln y = C$$

note

$$\int x e^{x^2} dx = 0$$

$\frac{d}{dx} x^2 = 2x$
 $\frac{d}{dx} e^{x^2} = 2x e^{x^2}$
 $\frac{1}{2x} \frac{d}{dx} e^{x^2} = e^{x^2}$
 $\frac{1}{4x^2} \frac{d}{dx} e^{x^2} = \frac{e^{x^2}}{2x}$

$$\text{ex 5: } \frac{dy}{dx} + p(x)y = 0$$

$$\int \frac{1}{y} dy = - \int p(x) dx$$

$$\Rightarrow y = C e^{-p(x)}$$

المتكامل

① no

②: Can be Separable

$$y' = f''(ax+by+c)$$

$$\text{Put } u = ax+by+c \quad \Rightarrow u' = a+by'$$

$$\Rightarrow y' = \frac{u'-a}{b}$$

$$\Rightarrow y' = f''(ax+by+c)$$

$$\therefore \frac{u'-a}{b} = f''(u)$$

$$\Rightarrow u' = b f''(u) + a$$

$$\Rightarrow \frac{du}{dx} = b f''(u) + a$$

$$u = ax+by+c$$

$$\Rightarrow \int \frac{1}{df''(u)+a} du = \int dx$$

المتكامل

ex: 1 $y' = (x+y+1)^2$

(Solution)

⇒ Can be Sep.

put $u = x+y+1$

$y' = u^2$

⇒ $\frac{1}{u^2+1} du = dx$

$\therefore \frac{1}{u^2+1}$ is of the form $\frac{1}{u^2+a^2}$ which is integrable

$\therefore \int \frac{1}{u^2+1} du = \int dx$

$\therefore \tan^{-1} u = x + C$

وحيث اننا لم نجد الاصل بدل ما نوجد في القاعده

$u = x+y+1$

$u' = 1 + y'$

$\therefore y' = u - 1$

$$\frac{\sin}{\cos} = \frac{\cos}{\sin} = \frac{1}{\cos}$$

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ex2: $y' = \tan^2(x+y+6)$ \cos, \sin

$$\therefore y' = \tan^2(u)$$

variable separable

$$\therefore \int \frac{du}{\tan^2 u + 1} = \int dx$$

$$\therefore \int \frac{du}{\sec^2 u} = \int dx \quad \therefore \int \cos^2 u \, du = \int dx$$

$$\cos^2 u = \frac{1}{2} + \frac{1}{2} \cos(2u)$$

$$\therefore \int \left(\frac{1}{2} + \frac{1}{2} \cos(2u) \right) du = \int dx$$

$$\therefore \frac{1}{2} u + \frac{1}{4} \sin 2u = x + C$$

put $u = x + y + 6$

~~ex~~ 3 Linear method

$$y' = P(x)y = Q(x)$$

Sol

$$Q(x) = 0 \quad \text{N.P.M.P.}$$

$$\therefore y' + P(x)y = 0$$

$$\therefore \frac{dy}{dx} = -P(x)y \quad \therefore \int \frac{1}{y} dy = \int -P(x) dx$$

$$\therefore y = e^{-\int P(x) dx} \cdot C$$

$$\therefore y e^{\int P(x) dx} = C$$

$$\frac{d}{dx} (y \cdot e^{\int P(x) dx}) = 0 \rightarrow \textcircled{1}$$

derivative

$$y \cdot P(x) e^{\int P(x) dx} + y' e^{\int P(x) dx} = 0$$

$$= (y' + P(x) y) e^{\int P(x) dx} = 0 \rightarrow 2$$

a-alla, well, ma ← also

$$\Rightarrow [y' + P(x) y] e^{\int P(x) dx} = Q e^{\int P(x) dx}$$

derivative

$$\Rightarrow \int \frac{d}{dx} (y \cdot e^{\int P(x) dx}) = \int Q \cdot e^{\int P(x) dx}$$

$$\Rightarrow y e^{\int P(x) dx} = \left[\int Q e^{\int P(x) dx} + C \right]$$

$$y = \frac{1}{e^{\int P(x) dx}} \left[\int Q e^{\int P(x) dx} + C \right]$$

$$\text{Put } e^{\int P(x) dx} = I$$

$$\therefore y = \frac{1}{I} \left(\int Q I dx + C \right)$$

ex:1 $y' + \left(\frac{-3}{x}\right) y = 1$

Sol

$$Q = 1$$
$$P(x) = \frac{-3}{x}$$

$$\rightarrow I = e^{\int \left(\frac{-3}{x}\right) dx} = e^{-3 \ln x} = e^{\ln(x)^{-3}} = \frac{1}{x^3}$$

$$y = \frac{1}{I} \left[\int \left(1 \times \frac{1}{x^3}\right) dx + C \right]$$

$$y = x^3 \left[\frac{-x^{-2}}{2} + C \right] = \frac{-x}{2} + Cx^3$$

ex:2 $\tan x \frac{dy}{dx} + y = \sin^2 x$ $\div \frac{\cos x}{\sin x}$

$$\therefore y' + \frac{\cos x}{\sin x} y = \sin x \cos x$$

$$\Rightarrow P(x) = \frac{\cos x}{\sin x}$$

$$\rightarrow Q(x) = \sin x \cos x$$

$$\therefore I = e^{\int \left(\frac{\cos x}{\sin x}\right) dx} = e^{\ln(\sin x)} = \sin x$$

$$y = \frac{1}{I} \left[\int (\sin x \cos x) (\sin x) dx + C \right]$$
$$= \frac{1}{\sin x} \left[\int (\sin^2 x \cos x) dx + C \right]$$

$$y = \frac{1}{\sin x} \left[\frac{\sin^3 x}{3} + C \right]$$

ex1: $y' + \tan y = \sin x$
Solution

$$P(x) = \frac{\sin x}{\cos x}$$

$$Q(x) = \sin x$$

$$I = \int P(x) dx$$

$$I = e^{-\int \frac{\sin x}{\cos x} dx} = e^{\ln(\cos x)} = \frac{1}{\cos x}$$

$$y = \frac{1}{\frac{1}{\cos x}} \left[- \int \frac{\sin x}{\cos x} dx + c \right]$$

$$= y = \cos x \left[\ln(\cos x) + c \right] \quad \#$$

ex2: $x \ln x \ln \ln x \quad y' + y = \left(1 + \frac{2}{x}\right) \ln x$

Final

Solution

$$y' + \frac{1}{x \ln x \ln \ln x} y = \frac{\left(1 + \frac{2}{x}\right)}{x \ln \ln x}$$

$$\Rightarrow P(x) = \frac{1}{x \ln x \ln \ln x}$$

$$Q(x) = \frac{1 + \frac{2}{x}}{x \ln \ln x}$$

$$I = e^{\int \frac{1}{x \ln x \ln \ln x} dx} = e^{\ln \ln \ln x} = \underline{\underline{\ln \ln x}}$$

$$y = \frac{1}{\ln \ln x} \left[\int \frac{1+2/x}{x \ln x} dx + C \right]$$

$$y = \frac{1}{\ln \ln x} \left[\int \left(\frac{1}{x} + \frac{2}{x^2} \right) dx + C \right]$$

$$= \frac{1}{\ln \ln x} \left[\ln x + \frac{-2}{x} + C \right] \quad \times$$

[4] Can be linear :- "Bernoulli eq"

$$y' + P(x)y = Q(x)y^n$$

$$y^n + \text{power} \quad (\text{Sol})$$

$$\Rightarrow \frac{1}{y^n} y' + P(x) y^{1-n} = Q(x) \rightarrow *$$

$$\Rightarrow \text{put } Z = y^{1-n} \rightarrow \textcircled{v}$$

$$\therefore Z' = (1-n) y^{-n} y'$$

$$\therefore Z' = (1-n) y^{-n} y' \Rightarrow \frac{Z'}{1-n} = \frac{y'}{y^n} \rightarrow *$$

* use consistency

$$\Rightarrow \therefore \frac{Z'}{1-n} + P(x) Z = Q(x)$$

$$\therefore Z' + (1-n) P(x) Z = (1-n) Q(x)$$

$$Z' + (1-n) P(x) Z = (1-n) Q(x)$$

$$\Rightarrow \therefore I = e^{\int (1-n) P(x) dx}$$

$$\therefore y^{1-n} = Z = \frac{1}{I} \left[\int I (1-n) Q(x) dx + C \right]$$

ex 1: $(1+x^2)y' - 2xy = x e^{-x} y^3$

Solution

$$\Rightarrow y' + \frac{-2x}{(1+x^2)} y = \frac{x e^{-x}}{(1+x^2)} y^3$$

$$P(x) = \frac{-2x}{1+x^2}$$

$$Q(x) = \frac{x e^{-x}}{1+x^2}$$

$$\Rightarrow I = e^{\int \frac{-2x}{1+x^2} (1-3) dx} = e^{2 \int \frac{2x}{1+x^2} dx}$$

$$I = e^{\ln(1+x^2)^2} = (1+x^2)^2$$

$$\Rightarrow Z = \frac{1}{(1+x^2)^2} \left[\int 2(1+x^2)^2 \cdot \frac{x e^{-x}}{(1+x^2)} dx + C \right]$$

$$Z = \frac{-2}{(1+x^2)^2} \left[\int (1+x) x e^{-x} dx + C \right]$$

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$$Z = \frac{-2}{(1+x^2)^2} \left[\int (x e^{-x} + x^3 e^{-x}) dx + c \right]$$

$$Z = \downarrow \left\{ -e^{-x} + 0 \right\}$$

$$\begin{array}{r} x^3 \quad e^{-x} \\ -3x^2 \quad \downarrow \quad -e^{-x} \\ +6x \quad \downarrow \quad +e^{-x} \\ -6 \quad \downarrow \quad -e^{-x} \\ -0 \quad \downarrow \quad +e^{-x} \end{array}$$

[5]: Homogeneous Method

From $M(x,y)dx + N(x,y)dy = 0$

$$\Rightarrow \text{Put } y = vx \quad v = \frac{y}{x}$$

$$y' = v + x \frac{dv}{dx}$$

$$x=1 \quad x y^2 y' = x^3 + y^3$$

$$f(x,y) = x y^2$$

$$\therefore f(\lambda x, \lambda y) = \lambda x (\lambda y)^2$$

$$= \lambda x \lambda^2 y^2 = \lambda^3 x y^2$$

$\therefore f$ is Homogeneous. \times

$$= \frac{x^3 + y^3}{x y^2}$$

possible

$$v + x \frac{dv}{dx} = \frac{x^3 + v^3 x^3}{x v^2 x^2}$$

$$\text{if } x^2 + y^2 \\ = (dx)^2 + (dy)^2$$

$$= d^2 x^2 + d^2 y^2$$

$$= d^2 (x^2 + y^2)$$

$\therefore f$ is Homogeneous

$$V + x \frac{dv}{dx} = \frac{1 + v^3}{v^2}$$

$$\cancel{v} + x \frac{dv}{dx} = \frac{1}{v^2} + \cancel{v}$$

$$\therefore \int v^2 dv = \int \frac{1}{x} dx \quad \therefore \frac{v^3}{3} = \ln x + C$$

$$v = \frac{y}{x}$$

ex2 $y' = \frac{x^2 + y^2}{xy}$

$$V + x \frac{dv}{dx} = \frac{x^2 + v^2 x^2}{x v x} = \frac{1 + v^2}{v}$$

$$\cancel{v} + x \frac{dv}{dx} = \frac{1}{v} + \cancel{v}$$

$$\therefore \int v dv = \int \frac{1}{x} dx$$

$$\frac{v^2}{2} = \ln x + C$$